Module:3 (CH-7)

7.5 The Heat Conduction Problem

We consider a homogeneous rod of length l. The rod is sufficiently thin so that the heat is distributed equally over the cross section at time t. The surface of the rod is insulated, and therefore, there is no heat loss through the boundary. The temperature distribution of the rod is given by the solution of the initial boundary-value problem

$$\begin{aligned} u_t &= k u_{xx}, \quad 0 < x < l, \quad t > 0, \\ u(0,t) &= 0, \qquad\qquad\qquad t \geq 0, \\ u(l,t) &= 0, \qquad\qquad\qquad t \geq 0, \\ u(x,0) &= f(x), \quad 0 \leq x \leq l. \end{aligned} \tag{7.5.1}$$

If we assume a solution in the form

$$u(x,t) = X(x) T(t) \neq 0.$$

Equation (7.5.1) yields

$$X T' = k X'' T.$$

Thus, we have

$$\frac{X''}{X} = \frac{T'}{kT} = -\alpha^2,$$

where α is a positive constant. Hence, X and T must satisfy

$$X'' + \alpha^2 X = 0, \tag{7.5.2}$$

$$T' + \alpha^2 k T = 0. \tag{7.5.3}$$

From the boundary conditions, we have

$$u(0,t) = X(0) T(t) = 0, \qquad u(l,t) = X(l) T(t) = 0.$$

Thus,

$$X(0) = 0, \qquad X(l) = 0,$$

for an arbitrary function $T(t)$. Hence, we must solve the eigenvalue problem

$$X'' + \alpha^2 X = 0,$$
$$X(0) = 0, \qquad X(l) = 0.$$

The solution of equation (7.5.2) is

$$X(x) = A\cos\alpha x + B\sin\alpha x.$$

Since $X(0) = 0$, $A = 0$. To satisfy the second condition, we have

$$X(l) = B\sin\alpha l = 0.$$

Since $B = 0$ yields a trivial solution, we must have $B \neq 0$ and hence,

$$\sin\alpha l = 0.$$

Thus,

$$\alpha = \frac{n\pi}{l} \quad \text{for} \quad n = 1, 2, 3 \ldots.$$

Substituting these eigenvalues, we have

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right).$$

Next, we consider equation (7.5.3), namely,

$$T' + \alpha^2 kT = 0,$$

the solution of which is

$$T(t) = Ce^{-\alpha^2 kt}.$$

Substituting $\alpha = (n\pi/l)$, we have

$$T_n(t) = C_n e^{-(n\pi/l)^2 kt}.$$

Hence, the nontrivial solution of the heat equation which satisfies the two boundary conditions is

$$u_n(x,t) = X_n(x)T_n(t) = a_n\, e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, 3 \ldots,$$

where $a_n = B_n C_n$ is an arbitrary constant.

By the principle of superposition, we obtain a formal series solution as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t),$$
$$= \sum_{n=1}^{\infty} a_n\, e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right), \qquad (7.5.4)$$

which satisfies the initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right).$$

This holds true if $f(x)$ can be represented by a Fourier sine series with Fourier coefficients

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \qquad (7.5.5)$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\tau)\sin\left(\frac{n\pi\tau}{l}\right)d\tau\right]e^{-(n\pi/l)^2kt}\sin\left(\frac{n\pi x}{l}\right) \quad (7.5.6)$$

is the formal series solution of the heat conduction problem.

Example 7.5.1. (a) Suppose the initial temperature distribution is $f(x) = x(l-x)$. Then, from equation (7.5.5), we have

$$a_n = \frac{8l^2}{n^3\pi^3}, \qquad n = 1, 3, 5, \ldots.$$

Thus, the solution is

$$u(x,t) = \left(\frac{8l^2}{\pi^3}\right)\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}e^{-(n\pi/l)^2kt}\sin\left(\frac{n\pi x}{l}\right).$$

(b) Suppose the temperature at one end of the rod is held constant, that is,

$$u(l,t) = u_0, \qquad t \geq 0.$$

The problem here is

$$\begin{aligned} u_t &= k\,u_{xx}, \quad 0<x<l, \quad t>0, \\ u(0,t) &= 0, \qquad u(l,t) = u_0, \qquad (7.5.7) \\ u(x,0) &= f(x), \quad 0<x<l. \end{aligned}$$

Let

$$u(x,t) = v(x,t) + \frac{u_0 x}{l}.$$

Substitution of $u(x,t)$ in equations (7.5.7) yields

$$\begin{aligned} v_t &= k\,v_{xx}, \quad 0<x<l, \quad t>0, \\ v(0,t) &= 0, \qquad v(l,t) = 0, \\ v(x,0) &= f(x) - \frac{u_0 x}{l}, \quad 0<x<l. \end{aligned}$$

Hence, with the knowledge of solution (7.5.6), we obtain the solution

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l\left(f(\tau)-\frac{u_0\tau}{l}\right)\sin\left(\frac{n\pi\tau}{l}\right)d\tau\right]e^{-(n\pi/l)^2kt}\sin\left(\frac{n\pi x}{l}\right) + \left(\frac{u_0 x}{l}\right). \quad (7.5.8)$$

Theorem 7.6.1. *(Uniqueness Theorem) Let $u(x,t)$ be a continuously differentiable function. If $u(x,t)$ satisfies the differential equation*

$$u_t = k\, u_{xx}, \quad 0 < x < l, \quad t > 0,$$

the initial conditions

$$u(x,0) = f(x), \quad 0 \le x \le l,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \ge 0,$$

then, the solution is unique.

Proof. Suppose that there are two distinct solutions $u_1(x,t)$ and $u_2(x,t)$. Let

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

Then,

$$\begin{aligned} v_t &= k\, v_{xx}, & 0 < x < l, && t > 0, \\ v(0,t) &= 0, & v(l,t) = 0, && t \ge 0, \\ v(x,0) &= 0, & 0 \le x \le l, && \end{aligned} \tag{7.6.3}$$

Consider the function defined by the integral

$$J(t) = \frac{1}{2k} \int_0^l v^2 dx.$$

Differentiating with respect to t, we have

$$J'(t) = \frac{1}{k} \int_0^l v v_t dx = \int_0^l v v_{xx} dx,$$

by virtue of equation (7.6.3). Integrating by parts, we have

$$\int_0^l v v_{xx} dx = [v v_x]_0^l - \int_0^l v_x^2 dx.$$

Since $v(0,t) = v(l,t) = 0$,

$$J'(t) = -\int_0^l v_x^2\, dx \le 0.$$

From the condition $v(x,0) = 0$, we have $J(0) = 0$. This condition and $J'(t) \le 0$ implies that $J(t)$ is a nonincreasing function of t. Thus,

$$J(t) \le 0.$$

But by definition of $J(t)$,

$$J(t) \ge 0.$$

Hence,

$$J(t) = 0, \quad \text{for } t \ge 0.$$

Since $v(x,t)$ is continuous, $J(t) = 0$ implies

$$v(x,t) = 0$$

in $0 \le x \le l$, $t \ge 0$. Therefore, $u_1 = u_2$ and the solution is unique.

Note: solve question 15 all parts from exercise