Module : 2
chapter 7

7.3 The Vibrating String Problem

As a first example, we shall consider the problem of a vibrating string of constant tension T^* and density ρ with $c^2 = T^*/\rho$ stretched along the x-axis from 0 to l, fixed at its end points. We have seen in Chapter 5 that the problem is given by

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < l, \qquad t > 0, \qquad (7.3.1)$$

$$u(x, 0) = f(x), \qquad 0 \le x \le l, \qquad (7.3.2)$$

$$u_t(x, 0) = g(x), \qquad 0 \le x \le l, \qquad (7.3.3)$$

$$u(0, t) = 0, \qquad t \ge 0, \qquad (7.3.4)$$

$$u(l, t) = 0, \qquad t \ge 0, \qquad (7.3.5)$$

where f and g are the initial displacement and initial velocity respectively.

By the method of separation of variables, we assume a solution in the form

$$u(x,t) = X(x)T(t) \neq 0. \tag{7.3.6}$$

If we substitute equation (7.3.6) into equation (7.3.1), we obtain

$$XT'' = c^2 X'' T,$$

and hence,

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}, \tag{7.3.7}$$

whenever $XT \neq 0$. Since the left side of equation (7.3.7) is independent of t and the right side is independent of x, we must have

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda,$$

where λ is a separation constant. Thus,

$$X'' - \lambda X = 0, \tag{7.3.8}$$
$$T'' - \lambda c^2 T = 0. \tag{7.3.9}$$

We now separate the boundary conditions. From equations (7.3.4) and (7.3.6), we obtain

$$u(0,t) = X(0)T(t) = 0.$$

We know that $T(t) \neq 0$ for all values of t, therefore,

$$X(0) = 0. \tag{7.3.10}$$

In a similar manner, boundary condition (7.3.5) implies

$$X(l) = 0. \tag{7.3.11}$$

To determine $X(x)$ we first solve the *eigenvalue problem* (eigenvalue problems are also treated in Chapter 8)

$$X'' - \lambda X = 0, \qquad X(0) = 0, \qquad X(l) = 0. \tag{7.3.12}$$

We look for values of λ which gives us nontrivial solutions. We consider three possible cases

$$\lambda > 0, \qquad \lambda = 0, \qquad \lambda < 0.$$

Case 1. $\lambda > 0$. The general solution in this case is of the form

$$X(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x}$$

where A and B are arbitrary constants. To satisfy the boundary conditions, we must have

$$A + B = 0, \qquad Ae^{-\sqrt{\lambda}l} + Be^{\sqrt{\lambda}l} = 0. \tag{7.3.13}$$

We see that the determinant of the system (7.3.13) is different from zero. Consequently, A and B must both be zero, and hence, the general solution $X(x)$ is identically zero. The solution is trivial and hence, is no interest.

Case 2. $\lambda = 0$. Here, the general solution is

$$X(x) = A + Bx.$$

Applying the boundary conditions, we have

$$A = 0, \qquad A + Bl = 0.$$

Hence $A = B = 0$. The solution is thus identically zero.

Case 3. $\lambda < 0$. In this case, the general solution assumes the form

$$X(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x.$$

From the condition $X(0) = 0$, we obtain $A = 0$. The condition $X(l) = 0$ gives

$$B\sin\sqrt{-\lambda}\,l = 0.$$

If $B = 0$, the solution is trivial. For nontrivial solutions, $B \neq 0$, hence,

$$\sin\sqrt{-\lambda}\,l = 0.$$

This equation is satisfied when

$$\sqrt{-\lambda}\,l = n\pi \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

or

$$-\lambda_n = (n\pi/l)^2. \tag{7.3.14}$$

For this infinite set of discrete values of λ, the problem has a nontrivial solution. These values of λ_n are called the *eigenvalues* of the problem, and the functions

$$\sin(n\pi/l)\,x, \qquad n = 1, 2, 3, \ldots$$

are the corresponding *eigenfunctions*.

We note that it is not necessary to consider negative values of n since

$$\sin(-n)\,\pi x/l = -\sin n\pi x/l.$$

No new solution is obtained in this way.

The solutions of problems (7.3.12) are, therefore,

$$X_n(x) = B_n \sin(n\pi x/l). \tag{7.3.15}$$

For $\lambda = \lambda_n$, the general solution of equation (7.3.9) may be written in the form

$$T_n(t) = C_n \cos\left(\frac{n\pi c}{l}\right) t + D_n \sin\left(\frac{n\pi c}{l}\right) t, \tag{7.3.16}$$

where C_n and D_n are arbitrary constants.

Thus, the functions

$$u_n(x,t) = X_n(x) T_n(t) = \left(a_n \cos\frac{n\pi c}{l} t + b_n \sin\frac{n\pi c}{l} t\right) \sin\left(\frac{n\pi x}{l}\right) \tag{7.3.17}$$

satisfy equation (7.3.1) and the boundary conditions (7.3.4) and (7.3.5), where $a_n = B_n C_n$ and $b_n = B_n D_n$.

Since equation (7.3.1) is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi c}{l} t + b_n \sin\frac{n\pi c}{l} t\right) \sin\left(\frac{n\pi x}{l}\right) \tag{7.3.18}$$

is also a solution, provided it converges and is twice continuously differentiable with respect to x and t. Since each term of the series satisfies the boundary conditions (7.3.4) and (7.3.5), the series satisfies these conditions. There remain two more initial conditions to be satisfied. From these conditions, we shall determine the constants a_n and b_n.

First we differentiate the series (7.3.18) with respect to t. We have

$$u_t = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \left(-a_n \sin\frac{n\pi c}{l} t + b_n \cos\frac{n\pi c}{l} t\right) \sin\left(\frac{n\pi x}{l}\right). \tag{7.3.19}$$

Then applying the initial conditions (7.3.2) and (7.3.3), we obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right), \tag{7.3.20}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right). \tag{7.3.21}$$

These equations will be satisfied if $f(x)$ and $g(x)$ can be represented by Fourier sine series. The coefficients are given by

$$a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \qquad b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx, \tag{7.3.22ab}$$

Note: Solve question 1,2 and 3 of exercise

Example 7.3.1. The Plucked String of length l

As a special case of the problem just treated, consider a stretched string fixed at both ends. Suppose the string is raised to a height h at $x = a$ and then released. The string will oscillate freely. The initial conditions, as shown in Figure 7.3.2, may be written

$$u(x,0) = f(x) = \begin{cases} hx/a, & 0 \le x \le a \\ h(l-x)/(l-a), & a \le x \le l. \end{cases}$$

Since $g(x) = 0$, the coefficients b_n are identically equal to zero. The coefficients a_n, according to equation (7.3.22a), are given by

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2}{l}\int_0^a \frac{hx}{a}\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_a^l \frac{h(l-x)}{(l-a)}\sin\left(\frac{n\pi x}{l}\right)dx. \end{aligned}$$

Integration and simplification yields

$$a_n = \frac{2hl^2}{\pi^2 a(l-a)}\frac{1}{n^2}\sin\left(\frac{n\pi a}{l}\right).$$

Thus, the displacement of the plucked string is

$$u(x,t) = \frac{2hl^2}{\pi^2 a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi c}{l}\right)t.$$

Example 7.3.2. The struck string of length l

Here, we consider the string with no initial displacement. Let the string be struck at $x = a$ so that the initial velocity is given by

$$u_t(x,0) = \begin{cases} \frac{v_0}{a}x, & 0 \le x \le a \\ v_0(l-x)/(l-a), & a \le x \le l \end{cases}.$$

Since $u(x,0) = 0$, we have $a_n = 0$. By applying equation (7.3.22b), we find that

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_0^a \frac{v_0}{a}x\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{n\pi c}\int_a^l v_0\frac{(l-x)}{(l-a)}\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2v_0 l^3}{\pi^3 ca(l-a)}\frac{1}{n^3}\sin\left(\frac{n\pi a}{l}\right). \end{aligned}$$

Hence, the displacement of the struck string is

$$u(x,t) = \frac{2v_0 l^3}{\pi^3 ca(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi c}{l}\right)t.$$

Theorem 7.4.2. *(Uniqueness Theorem) There exists at most one solution of the wave equation*

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0,$$

satisfying the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq l,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \geq 0,$$

where $u(x,t)$ is a twice continuously differentiable function with respect to both x and t.

Proof. Suppose that there are two solutions u_1 and u_2 and let $v = u_1 - u_2$. It can readily be seen that $v(x,t)$ is the solution of the problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, && 0 < x < l, && t > 0, \\ v(0,t) &= 0, && && t \geq 0, \\ v(l,t) &= 0, && && t \geq 0, \\ v(x,0) &= 0, && 0 \leq x \leq l, && \\ v_t(x,0) &= 0, && 0 \leq x \leq l. && \end{aligned}$$

We shall prove that the function $v(x,t)$ is identically zero. To do so, consider the energy integral

$$E(t) = \frac{1}{2}\int_0^l \left(c^2 v_x^2 + v_t^2\right) dx \tag{7.4.10}$$

which physically represents the total energy of the vibrating string at time t.

Since the function $v(x,t)$ is twice continuously differentiable, we differentiate $E(t)$ with respect to t. Thus,

$$\frac{dE}{dt} = \int_0^l \left(c^2 v_x v_{xt} + v_t v_{tt}\right) dx. \tag{7.4.11}$$

Integrating the first integral in (7.4.11) by parts, we have

$$\int_0^l c^2 v_x v_{xt} dx = \left[c^2 v_x v_t\right]_0^l - \int_0^l c^2 v_t v_{xx} dx.$$

But from the condition $v(0,t) = 0$ we have $v_t(0,t) = 0$, and similarly, $v_t(l,t) = 0$ for $x = l$. Hence, the expression in the square brackets vanishes, and equation (7.4.11) becomes

$$\frac{dE}{dt} = \int_0^l v_t \left(v_{tt} - c^2 v_{xx}\right) dx. \tag{7.4.12}$$

Since $v_{tt} - c^2 v_{xx} = 0$, equation (7.4.12) reduces to

$$\frac{dE}{dt} = 0$$

which means

$$E(t) = \text{constant} = C.$$

Since $v(x,0) = 0$ we have $v_x(x,0) = 0$. Taking into account the condition $v_t(x,0) = 0$, we evaluate C to obtain

$$E(0) = C = \frac{1}{2}\int_0^l \left[c^2 v_x^2 + v_t^2\right]_{t=0} dx = 0.$$

This implies that $E(t) = 0$ which can happen only when $v_x = 0$ and $v_t = 0$ for $t > 0$. To satisfy both of these conditions, we must have $v(x,t) =$ constant. Employing the condition $v(x,0) = 0$, we then find $v(x,t) = 0$. Therefore, $u_1(x,t) = u_2(x,t)$ and the solution $u(x,t)$ is unique.