Lecture 2 – Maxwell's Equations, EM wave

CHARGE AND ENERGY

The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics. But I want to begin by reviewing the conservation of *charge*, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure—that's **global** conservation of charge. But **local** conservation of charge is a much stronger statement: If the charge in some region changes, then exactly that amount of charge must have passed in or out through the surface. The tiger can't simply rematerialize outside the cage; if it got from inside to outside it must have slipped through a hole in the fence.

Formally, the charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau, \tag{8.1}$$

and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$, so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{8.2}$$

Using Eq. 8.1 to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau, \tag{8.3}$$

and since this is true for *any* volume, it follows that

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.} \tag{8.4}$$

This is the Equation of Continuity (Mathematical Statement of local conservation of Charge)

Poynting's Theorem

Work required to assemble a static distribution of charges is given as

$$W_e = \frac{\epsilon_0}{2} \int E^2 \, d\tau,$$

Already solved in Electrostatics

Work required to get the currents going on against the back emf is

$$W_m = \frac{1}{2\mu_0} \int B^2 \, d\tau,$$

Already solved in Magnetostatics

Therefore total energy stored in electromagnetic field s is given as

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \qquad (8.5)$$

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges, in the interval dt? According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F} \cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = q\mathbf{E} \cdot \mathbf{v}\,dt.$$

In terms of the charge and current densities, $q \to \rho d\tau$ and $\rho \mathbf{v} \to \mathbf{J}$,

So the rate at which work is done on all charges in a volume γ is

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau. \qquad (8.6)$$

Evidently $\mathbf{E} \cdot \mathbf{J}$ is the work done per unit time, per unit volume—which is to say, the *power* delivered per unit volume. We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

From product rule 6,

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$), it follows that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}).$$

Meanwhile,

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{8.7}$$

so

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}). \tag{8.8}$$

Putting this into Eq. 8.6, and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}, \tag{8.9}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, $\int u\, d\tau$ (Eq. 8.5). The second term evidently represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that *the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.*

The *energy per unit time, per unit area*, transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).} \tag{8.10}$$

Maxwell's Equations

In the last section we put the finishing touches on Maxwell's equations:

(i)	$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$	(Gauss's law),
(ii)	$\nabla \cdot \mathbf{B} = 0$	(no name),
(iii)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(Faraday's law),
(iv)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	(Ampère's law with Maxwell's correction).

Physical Significance of Maxwell's Equations

By means of Gauss and Stoke's theorem we can put the field equations in integral form of hence obtain their physical significance

1. Maxwell's first equation is

$$\nabla. D = \rho.$$

Integrating this over an arbitrary volume V we get

$$\int_V \nabla.D \, dV = \int_V \rho \, dV.$$

But from Gauss Theorem, we get

$$\int_S D.dS = \int_V \rho \, dV = q$$

Here, q is the net charge contained in volume V. S is the surface bounding volume V. Therefore,

Maxwell's first equation signifies that:
The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

2. Maxwell's second equations is

$$\nabla . B = 0$$

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla . B = 0.$$

Using Gauss divergence theorem to change volume integral into surface integral, we get

$$\int_S B.dS = 0.$$

Maxwell's second equation signifies that:
The total outward flux of magnetic induction B through any closed surface S is equal to zero.

3. Maxwell's third equation is

$$\nabla \times E = - \partial B/\partial t \;.\; dS$$

Converting the surface integral of left hand side into line integral by Stoke's theorem, we get

$$\int_C E.\, dl = - \int_S \partial B/\partial t.\, dS.$$

Maxwell's third equation signifies that:
The electromotive force (e.m.f. e = $\int_C$ E.dl) around a closed path is equal to negative rate of change of magnetic flux linked with the path (since magnetic flux Φ = $\int_s$ B.dS).

4. Maxwell's fourth equation is

$$\nabla \times H = J + \partial D/\partial t$$

Taking surface integral over surface S bounded by curve C, we obtain

$$\int_s \nabla \times H.\, dS = \int_s (J + \partial D/\partial t)\, dS$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral, we get

$$\int_c H.dl = \int_s (J + \partial D/\partial t).dS$$

Maxwell's fourth equation signifies that:
The magneto motive force (m.m.f. = $\int$c H. dl) around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.