Lecture 4

(Poisson Distribution)

Course	**B.Sc. (H) Physics**
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Teacher's Name	**Ms Sonia Yogi**
Department	**Physics and Electronics, Hansraj College DU**

0.1 Important points regarding Poisson Distribution

(a). Poisson distribution is a special case of Binomial distribution.

(b). Number of trials is infinite $(n \to \infty)$ and probability of success, $p \to 0$ such that $n\,p = \lambda$ is finite.

(c). In each trial, probability of success is *p* and probability of failure is $q(= 1 - p)$.

(d). Each trial is independent and is done in identical conditions and circumstances.

(e). Probability of *r* success is given by

$$P(r) = \frac{\lambda^r\, e^{-\lambda}}{r!}; \qquad \lambda = n\,p$$

0.2 Mean of Poisson Ditribution

Mean of Poisson distribution is defined as

$$\begin{aligned}
\bar{x} &= \frac{\sum_{r=0}^{n} r\,P(r)}{\sum_{r=0}^{n} P(r)} \\
&= \frac{\sum_{r=0}^{\infty} r\,P(r)}{\sum_{r=0}^{\infty} P(r)} && [\because n \to \infty] \\
&= \sum_{r=0}^{\infty} r\,P(r) && \left[\because \sum_{r=0}^{\infty} P(r) = \sum_{r=0}^{\infty} \frac{\lambda^r\, e^{-\lambda}}{r!} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = e^{-\lambda}\, e^{\lambda} = 1\right] \\
&= \sum_{r=0}^{\infty} r\,\frac{\lambda^r\, e^{-\lambda}}{r!} && [\because P(r) = \frac{\lambda^r\, e^{-\lambda}}{r!}] \\
&= \lambda\, e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!} \\
&= \lambda\, e^{-\lambda}\, e^{\lambda} = \lambda = n\,p
\end{aligned}$$

$\bar{x}$ is also called expectation of random variable X i.e., $\bar{x} = E(X)$.

Note: $E(X)$ is also known as first moment, denoted by μ_1.

0.3 Second Moment of Binomial Distribution

Second Moment of Poisson distribution is given by

$$
\begin{aligned}
\bar{x} &= \frac{\sum_{r=0}^{n} r^2\, P(r)}{\sum_{r=0}^{n} P(r)} \\
&= \frac{\sum_{r=0}^{\infty} r^2\, P(r)}{\sum_{r=0}^{\infty} P(r)} && [\because n \to \infty] \\
&= \sum_{r=0}^{\infty} r^2\, P(r) && \left[\because \sum_{r=0}^{\infty} P(r) = \sum_{r=0}^{\infty} \frac{\lambda^r\, e^{-\lambda}}{r!} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = e^{-\lambda}\, e^{\lambda} = 1\right] \\
&= \sum_{r=0}^{\infty} [r(r-1) + r]\, P(r) \\
&= \sum_{r=0}^{\infty} r(r-1)\, P(r) + \sum_{r=0}^{\infty} r\, P(r) \\
&= \sum_{r=0}^{\infty} r(r-1)\, P(r) + \lambda && \text{[From first moment]} \\
&= \sum_{r=0}^{\infty} r(r-1)\, \frac{\lambda^r\, e^{-\lambda}}{r!} + \lambda && [\because P(r) = \frac{\lambda^r\, e^{-\lambda}}{r!}] \\
&= \lambda^2\, e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} + \lambda \\
&= \lambda^2\, e^{-\lambda}\, e^{\lambda} + \lambda \\
&= \lambda(\lambda + 1)
\end{aligned}
$$

$E(X^2)$ is also known as second moment, denoted by μ_2.

0.4 Variance of Poisson Distribution

Variance of Poisson distribution is given by

$$
\begin{aligned}
\sigma^2 &= \mu_2 - \mu_1^2 \\
&= \lambda(\lambda + 1) - \lambda^2 \\
&= \lambda
\end{aligned}
$$

0.5 Standard Deviation of Poisson Distribution

Standard Deviation of Poisson Distribution is given by

$$\sigma = \sqrt{\sigma^2} = \sqrt{\lambda} = \sqrt{n\,p}$$

0.6 Summary (Binomial Distribution)

Mean or Expectation	$E = \lambda = n\,p$
Variance	$\sigma^2 = \lambda = n\,p$
Standard Deviation	$\sigma = \sqrt{n\,p}$

0.7 Examples

Example 1. Show that if p is small and n is large, then the Binomial distribution $B(n, p)$ is approximated by the Poisson distribution $P(\lambda)$, where $\lambda = n\,p$.

Solution In Binomial distribution, we have

$$\begin{aligned}
P(r) &= \binom{n}{r} p^r\, q^{n-r};\ q = 1-p \\
&= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\, p^r\,(1-p)^{n-r} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\, \frac{\lambda^r}{n^r}\left(1-\frac{\lambda}{n}\right)^{n-r} \qquad [\text{putting } p = \lambda/n] \\
&= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r} \\
&= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\frac{\left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}
\end{aligned}$$

In the limit $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty} P(r) &= \frac{\lambda^r}{r!}(1-0)\,(1-0)\ldots(1-0)\frac{e^{-\lambda}}{(1-0)^r} \qquad \left[\because \lim_{n\to\pm\infty}\left(1+\frac{1}{n}\right)^n = e\right] \\
&= \frac{\lambda^r}{r!}e^{-\lambda}
\end{aligned}$$

i.e.,

$$\boxed{P(X = r) = \frac{\lambda^r\, e^{-\lambda}}{r!}}$$

Example 2. If X is a Poisson variate and $P[X = 0] = P[X = 1] = k$, show that $k = \frac{1}{e}$

Solution We have,

$$P[X = 0] = \frac{\lambda^0\, e^{-\lambda}}{0!} = e^{-\lambda} = k \implies \lambda = \ln\left(\frac{1}{k}\right)$$

Similarly,

$$P[X=1]=\frac{\lambda^1\,e^{-\lambda}}{1!}=\lambda\,e^{-\lambda}=k\implies\lambda\,k=k\implies\lambda=1$$

Therefore,

$$\ln\left(\frac{1}{k}\right)=1\implies\boxed{k=\frac{1}{e}}$$

Example 3. If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 2000 individuals (a) exactly 3, (b) more than 2 individuals will suffer a bad reaction.

Solution Here, the no. of trials $(n)=2000$ and the probability of success $(p)=0.001$ which gives $\lambda=n\,p=2000\times0.001=2$ and let X denotes the no. of individuals suffers a bad reaction.

(a) We have to compute $P[X=3]$,

$$\begin{aligned}P[X=3]&=\frac{2^3\,e^{-2}}{3!}\\&=\frac{4\,e^{-2}}{3}\approx0.18\end{aligned}$$

(b) We have to compute $P[X>2]$,

$$\begin{aligned}P[X>2]&=1-P[X=1]-P[X=2]\\&=1-\frac{2^1\,e^{-2}}{1!}-\frac{2^2\,e^{-2}}{2!}\\&=1-4\,e^{-2}\\&\approx1-0.541=0.459\end{aligned}$$

Example 4. Assume that the probability that a bomb dropped from an aeroplane will strike a certain target is $1/5$. If 6 bombs are dropped, find the probability that

(a). exactly 2 will strike the target and

(b). at least 2 will strike the target.

Solution Let X denotes the number of bombs that strike the target. Here, the no . of trials (n) is 6 and probability of success (p) is $1/5$ which gives $\lambda=n\times p=6\times1/5=6/5$.

(a) We have to compute $P[X=2]$,

$$\begin{aligned}P[X=2]&=\frac{(6/5)^2\,e^{-(6/5)}}{2!}\\&=\frac{36\,e^{-(6/5)}}{50}=\frac{36\times0.3012}{50}\approx0.217\end{aligned}$$

(b) We have to compute $P[X \geq 2]$,

$$\begin{aligned} P[X \geq 2] &= 1 - P[X=0] - P[X=1] \\ &= 1 - \frac{(6/5)^0\, e^{-(6/5)}}{0!} - \frac{(6/5)^1\, e^{-(6/5)}}{1!} \\ &= 1 - e^{-(6/5)} - \frac{6\, e^{-(6/5)}}{5} \\ &= 1 - \frac{11 \times 0.3012}{5} \\ &\approx 1 - 0.663 = 0.337 \end{aligned}$$

Example 5. A pair of dice is thrown 200 times. If getting a sum of 9 is considered a success, find the mean and the variance of the number of successes.

Solution Let X denotes the number that a sum of 9 occurs. Here, the number of trials (n) is 200 and the probability of success (p) is $4/36 = 1/9$ which gives $\lambda = n\,p = 200 \times 1/9 = 200/9$.

The mean is given by,

$$\mu = \lambda = 200/9.$$

The variance is given by,

$$\sigma^2 = \lambda = 200/9.$$

0.8 Problems

1. If X is the number of occurrences of the Poisson variate with mean λ, show that
$$P[X \geq n] - P[X \geq n+1] = P[X = n].$$

2. Prove that $P(r+1) = \frac{\lambda}{r+1} P(r); \qquad r = 0, 1, 2, 3, \ldots.$

3. Suppose 1% of the items made by a machine are defective. Find the probability that 3 or more items are defective in a sample of 100 items.

4. A bag contains 1 red and 7 white marbles. A marble is drawn from the bag, and its color is observed. Then the marble is put back into the bag and the contents are thoroughly mixed. Using (a) the binomial distribution, (b) the Poisson approximation to the binomial distribution, find the probability that in 8 such drawings, a red ball is selected exactly 3 times.

5. Suppose 220 misprints are distributed randomly throughout a book of 200 pages. Find the probability that a given page contains: (a) no misprints, (b) 1 misprint, (c) 2 misprints, (d) 2 or more misprints.

6. Suppose there is an average of 2 suicides per year per $50,000$ population. In a city of $100,000$, find the probability that in a given year the number of suicides is: (a) 0, (b) 1, (c) 2, (d) 2 or more.

7. If on an average 9 ships out of 10 arrive safely to a port, obtain mean and standard deviation of the number of ships arriving safely out of a total of 150 ships using

   (a) Poisson distribution

   (b) Binomial distribution

0.9 References

(a). Introduction to Probability and Statistics, 2016 by Seymour Lipschutz and John J. Schiller.

(b). Probability and Statistics, 3e by Murray R. Spiegel, John J. Schiller and R. Alu Srinivasan.